Published by Institute of Physics Publishing for SISSA
Received: *June 23, 2006*
Accepted: *August 16, 2006*
Published: *September 11, 2006*

Dynamical generation of fuzzy extra dimensions, dimensional reduction and symmetry breaking

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Abstract: We present a renormalizable 4-dimensional SU($\mathcal{N}$) gauge theory with a suitable multiplet of scalar fields, which dynamically develops extra dimensions in the form of a fuzzy sphere S^2_N. We explicitly find the tower of massive Kaluza-Klein modes consistent with an interpretation as gauge theory on $M^4 \times S^2$, the scalars being interpreted as gauge fields on S^2. The gauge group is broken dynamically, and the low-energy content of the model is determined. Depending on the parameters of the model the low-energy gauge group can be SU(n), or broken further to SU(n_1) $\times$ SU(n_2) $\times$ U(1), with mass scale determined by the size of the extra dimension.

Keywords: Matrix Models, Field Theories in Higher Dimensions, Non-Commutative Geometry, Spontaneous Symmetry Breaking.

 http://jhep.sissa.it/archive/papers/jhep092006026/jhep092006026.pdf

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1. Introduction

It is difficult to overestimate the relevance of the Kaluza-Klein programme of unification in higher dimensions. In this beautiful programme, higher dimensions are an input however, and the 4-dimensional theory has to be recovered. We here reverse the logic and see how a 4-dimensional gauge theory dynamically develops higher dimensions. The very concept of dimension therefore gets an extra, richer dynamical perspective. For pioneering work in that context see [1]. Furthermore, the Kaluza-Klein programme can now be pursued within the framework of a 4-dimensional field theory, which dynamically develops higher dimensions.

We present in this paper a simple field-theoretical model which realizes that idea. It is defined as a renormalizable $SU(\mathcal{N})$ gauge theory on 4-dimensional Minkowski space M^4, containing 3 scalars in the adjoint of $SU(\mathcal{N})$ that transform as vectors under an additional global $SO(3)$ symmetry with the most general renormalizable potential. We then show that the model dynamically develops fuzzy extra dimensions, more precisely a fuzzy sphere S^2_N. The appropriate interpretation is therefore as gauge theory on $M^4 \times S^2_N$. The low-energy effective action is that of a 4-dimensional gauge theory on M^4, whose gauge group and field content is dynamically determined by compactification and dimensional reduction on the internal sphere S^2_N. An interesting and quite rich pattern of spontaneous symmetry breaking appears, breaking the original SU($\mathcal{N}$) gauge symmetry down to much smaller and potentially quite interesting low-energy gauge groups. In particular, we find explicitly the tower of massive Kaluza-Klein states, which justifies the interpretation as a compactified higher-dimensional gauge theory. Nevertheless, the model is renormalizable.

A different mechanism of dynamically generating extra dimensions has been proposed some years ago in [1], known under the name of "deconstruction". In this context, renormalizable 4-dimensional asymptotically free gauge theories were considered with suitable Moose- or Quiver-type arrays of gauge groups and couplings, which develop a "lattice-like" fifth dimension. This idea attracted considerable interest. Our model is quite different, and very simple: The SU($\mathcal{N}$) gauge theory with 3 scalars ϕ_a in the adjoint and a global SO(3) symmetry is shown to develop fuzzy extra dimensions through a symmetry breaking mechanism.

Let us discuss some of the features of our model in more detail. The effective geometry, the symmetry breaking pattern and the low-energy gauge group are determined dynamically in terms of a few free parameters of the potential. We discuss in detail the two simplest possible vacua with gauge groups SU(n) and SU(n_1) $\times$ SU(n_2) $\times$ U(1). We find explicitly the tower of massive Kaluza-Klein modes corresponding to the effective geometry. The mass scale of these massive gauge bosons is determined by the size of the extra dimensions, which in turn depends on some logarithmically running coupling constants. In the case of the SU(n_1) $\times$ SU(n_2) $\times$ U(1) vacuum, we identify in particular massive gauge fields in the bifundamental, similar as in GUT models with an adjoint Higgs. Moreover, we also identify a candidate for a further symmetry breaking mechanism, which may lead to a low-energy content of the theory close to the standard model.

There is no problem in principle to add fermions to our model. In particular, we point out that in the vacua with low-energy gauge group SU(n_1) $\times$ SU(n_2) $\times$ U(1), the extra-dimensional sphere always carries a magnetic flux with nonzero monopole number. This is very interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions. However, this is a delicate issue and will be discussed in a forthcoming paper.

Perhaps the most remarkable aspect of our model is that the geometric interpretation and the corresponding low-energy degrees of freedom depend in a nontrivial way on the parameters of the model, which are running under the RG group. Therefore the massless degrees of freedom and their geometrical interpretation depend on the energy scale. In particular, the low-energy gauge group generically turns out to be SU(n_1) $\times$ SU(n_2) $\times$ U(1) or SU(n), while gauge groups which are products of more than two simple components (apart from U(1)) do not seem to occur in this model. Moreover, the values of n_1 and n_2 are determined dynamically, and may well be small such as 3 and 2. A full analysis of the hierarchy of all possible vacua and their symmetry breaking pattern is not trivial however, and will not be attempted in this paper. Here we restrict ourselves to establish the basic mechanisms and features of the model, and discuss in section 3 the two simplest cases (that we name "type 1" and "type 2" vacuum) in some detail. A more detailed analysis (in particular for the "type 3 vacuum") is left for future work.

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The idea to use fuzzy spaces for the extra dimensions is certainly not new. This work was motivated by a fuzzy coset space dimensional reduction (CSDR) scheme considered recently in [2 − 4], combined with lessons from the matrix-model approach to gauge theory on the fuzzy sphere [5, 6]. This leads in particular to a dynamical mechanism of determining the vacuum, SSB patterns and background fluxes. A somewhat similar model has been studied recently in [7, 8], which realizes deconstruction and a "twisted" compactification of an extra fuzzy sphere based on a supersymmetric gauge theory. Our model is different and does not require supersymmetry, leading to a much richer pattern of symmetry breaking and effective geometry. For other relevant work see e.g. [9].

The dynamical formation of fuzzy spaces found here is also related to recent work studying the emergence of stable submanifolds in modified IIB matrix models. In particular, previous studies based on actions for fuzzy gauge theory different from ours generically only gave results corresponding to U(1) or U(∞) gauge groups, see e.g. [10 − 12] and references therein. The dynamical generation of a nontrivial index on noncommutative spaces has also been observed in [13, 14] for different models.

Our mechanism may also be very interesting in the context of the recent observation [15] that extra dimensions are very desirable for the application of noncommutative field theory to particle physics. Other related recent work discussing the implications of the higher-dimensional point of view on symmetry breaking and Higgs masses can be found in [16 − 19]. These issues could now be discussed within a renormalizable framework.

2. The 4-dimensional action

We start with a SU($\mathcal{N}$) gauge theory on 4-dimensional Minkowski space M^4 with coordinates y^μ, $\mu = 0, 1, 2, 3$. The action under consideration is

$$\mathcal{S}_{YM} = \int d^4y\, Tr\, \left(\frac{1}{4g^2}\, F^\dagger_{\mu\nu} F_{\mu\nu} + (D_\mu\phi_a)^\dagger D_\mu\phi_a\right) - V(\phi) \tag{2.1}$$

where A_μ are $\mathfrak{su}(\mathcal{N})$-valued gauge fields, $D_\mu = \partial_\mu + [A_\mu, .]$, and

$$\phi_a = -\phi_a^\dagger, \qquad a = 1, 2, 3 \tag{2.2}$$

are 3 antihermitian scalars in the adjoint of SU($\mathcal{N}$),

$$\phi_a \to U^\dagger \phi_a U \tag{2.3}$$

where $U = \mathrm{U}(y) \in \mathrm{SU}(\mathcal{N})$. Furthermore, the ϕ_a transform as vectors of an additional global SO(3) symmetry. The potential $V(\phi)$ is taken to be the most general renormalizable action invariant under the above symmetries, which is

$$\begin{aligned} V(\phi) = {} & Tr\, \left(g_1\phi_a\phi_a\phi_b\phi_b + g_2\phi_a\phi_b\phi_a\phi_b - g_3\varepsilon_{abc}\phi_a\phi_b\phi_c + g_4\phi_a\phi_a\right) \\ & + \frac{g_5}{\mathcal{N}}\, Tr(\phi_a\phi_a)Tr(\phi_b\phi_b) + \frac{g_6}{\mathcal{N}} Tr(\phi_a\phi_b)Tr(\phi_a\phi_b) + g_7. \end{aligned} \tag{2.4}$$

This may not look very transparent at first sight, however it can be written in a very intuitive way. First, we make the scalars dimensionless by rescaling

$$\phi'_a = R\, \phi_a, \tag{2.5}$$

where R has dimension of length; we will usually suppress R since it can immediately be reinserted, and drop the prime from now on. Now observe that for a suitable choice of R,

$$R = \frac{2g_2}{g_3}, \tag{2.6}$$

the potential can be rewritten as

$$V(\phi) = Tr\left(a^2(\phi_a\phi_a + \tilde{b}\,\mathbb{1})^2 + c + \frac{1}{\tilde{g}^2}F^\dagger_{ab}F_{ab}\right) + \frac{h}{\mathcal{N}}g_{ab}g_{ab} \tag{2.7}$$

for suitable constants $a, b, c, \tilde{g}, h$, where

$$\begin{aligned} F_{ab} &= [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c = \varepsilon_{abc}F_c, \\ \tilde{b} &= b + \frac{d}{\mathcal{N}}Tr(\phi_a\phi_a), \\ g_{ab} &= Tr(\phi_a\phi_b). \end{aligned} \tag{2.8}$$

We will omit c from now. The potential is clearly positive definite provided

$$a^2 = g_1 + g_2 > 0, \qquad \frac{2}{\tilde{g}^2} = -g_2 > 0, \qquad h \geq 0, \tag{2.9}$$

which we assume from now on. Here $\tilde{b} = \tilde{b}(y)$ is a scalar, $g_{ab} = g_{ab}(y)$ is a symmetric tensor under the global SO(3), and $F_{ab} = F_{ab}(y)$ is a $\mathfrak{su}(\mathcal{N})$-valued antisymmetric tensor field which will be interpreted as field strength in some dynamically generated extra dimensions below. In this form, $V(\phi)$ looks like the action of Yang-Mills gauge theory on a fuzzy sphere in the matrix formulation [5, 6, 20, 21]. The presence of the first term $a^2(\phi_a\phi_a + \tilde{b})^2$ might seem strange at first, however we should not simply omit it since it would be reintroduced by renormalization. In fact it is necessary for the interpretation as YM action, and we will see that it is very welcome on physical grounds since it dynamically determines and stabilizes a vacuum, which can be interpreted as extra-dimensional fuzzy sphere. In particular, it removes unwanted flat directions.

Let us briefly comment on the RG flow of the various constants. Without attempting any precise computations here, we can see by looking at the potential (2.4) that g_4 will be quadratically divergent at one loop, while g_1 and g_2 are logarithmically divergent. Moreover, the only diagrams contributing to the coefficients g_5, g_6 of the "nonlocal" terms are nonplanar, and thus logarithmically divergent but suppressed by $\frac{1}{\mathcal{N}}$ compared to the other (planar) diagrams. This justifies the explicit factors $\frac{1}{\mathcal{N}}$ in (2.4) and (2.8). Finally, the only one-loop diagram contributing to g_3 is also logarithmically divergent. In terms of the constants in the potential (2.7), this implies that R, a, $\tilde{g}$, d and h are running logarithmically under the RG flux, while b and therefore $\tilde{b}$ is running quadratically. The gauge coupling g is of course logarithmically divergent and asymptotically free.

A full analysis of the RG flow of these parameters is complicated by the fact that the vacuum and the number of massive resp. massless degrees of freedom depends sensitively on the values of these parameters, as will be discussed below. This indicates that the RG flow of this model will have a rich and nontrivial structure, with different effective description at different energy scales.

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2.1 The minimum of the potential

Let us try to determine the minimum of the potential (2.7). This turns out to be a rather nontrivial task, and the answer depends crucially on the parameters in the potential.

For suitable values of the parameters in the potential, we can immediately write down the vacuum. Assume for simplicity $h = 0$ in (2.7) . Since $V(\phi) \geq 0$, the global minimum of the potential is certainly achieved if

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c \ = 0, \qquad -\phi_a\phi_a = \tilde{b}, \tag{2.10}$$

because then $V(\phi) = 0$. This implies that ϕ_a is a representation of SU(2), with prescribed Casimir[1] $\tilde{b}$. These equations may or may not have a solution, depending on the value of $\tilde{b}$. Assume first that $\tilde{b}$ coincides with the quadratic Casimir of a finite-dimensional irrep of SU(2),

$$\tilde{b} = C_2(N) = \frac{1}{4}(N^2 - 1) \tag{2.11}$$

for some $N \in \mathbb{N}$. If furthermore the dimension $\mathcal{N}$ of the matrices ϕ_a can be written as

$$\mathcal{N} = Nn, \tag{2.12}$$

then clearly the solution of (2.10) is given by

$$\phi_a = X_a^{(N)} \otimes \mathbb{1}_n \tag{2.13}$$

up to a gauge transformation, where $X_a^{(N)}$ denote the generator of the N-dimensional irrep of SU(2). This can be viewed as a special case of (2.15) below, consisting of n copies of the irrep (N) of SU(2).

For generic $\tilde{b}$, the equations (2.10) cannot be satisfied for finite-dimensional matrices ϕ_a. The exact vacuum (which certainly exists since the potential is positive definite) can in principle be found by solving the "vacuum equation" $\frac{\delta V}{\delta \phi_a} = 0$,

$$a^2\{\phi_a, \phi\cdot\phi + \tilde{b} + \frac{d}{\mathcal{N}}Tr(\phi\cdot\phi + \tilde{b})\} + \frac{2h}{\mathcal{N}}g_{ab}\phi_b + \frac{1}{\tilde{g}^2}\left(2[F_{ab}, \phi_b] + F_{bc}\varepsilon_{abc}\right) = 0 \tag{2.14}$$

where $\phi\cdot\phi = \phi_a\phi_a$. We note that all solutions under consideration will imply $g_{ab} = \frac{1}{3}\delta_{ab}Tr(\phi\cdot\phi)$, simplifying this expression.

The general solution of (2.14) is not known. However, it is easy to write down a large class of solutions: any decomposition of $\mathcal{N} = n_1N_1 + \cdots + n_hN_h$ into irreps of SU(2) with multiplicities n_i leads to a block-diagonal solution

$$\phi_a = diag\Big(\alpha_1\, X_a^{(N_1)}, \ldots, \alpha_k\, X_a^{(N_k)}\Big) \tag{2.15}$$

of the vacuum equations (2.14), where α_i are suitable constants which will be determined below. There are hence several possibilities for the true vacuum, i.e. the global minimum of the potential. Since the general solution is not known, we proceed by first determining the solution of the form (2.15) with minimal potential, and then discuss a possible solution of a different type ("type 3 vacuum").

[1]note that $-\phi\cdot\phi = \phi^\dagger\cdot\phi > 0$ since the fields are antihermitian

Type 1 vacuum. It is clear that the solution with minimal potential should satisfy (2.10) at least approximately. It is therefore plausible that the solution (2.15) with minimal potential contains only representations whose Casimirs are close to $\tilde{b}$. In particular, let N be the dimension of the irrep whose Casimir $C_2(N) \approx \tilde{b}$ is closest to $\tilde{b}$. If furthermore the dimensions match as $\mathcal{N} = Nn$, we expect that the vacuum is given by n copies of the irrep (N), which can be written as

$$\phi_a = \alpha\, X_a^{(N)} \otimes 1\!\!1_n. \tag{2.16}$$

This is a slight generalization of (2.13), with α being determined through the vacuum equations (2.14),

$$a^2(\alpha^2 C_2(N) - \tilde{b})(1+d) + \frac{h}{3}\alpha^2 C_2(N) - \frac{1}{\tilde{g}^2}(\alpha-1)(1-2\alpha) = 0 \tag{2.17}$$

A vacuum of the form (2.16) will be denoted as "type 1 vacuum". As we will explain in detail, it has a natural interpretation in terms of a dynamically generated extra-dimensional fuzzy sphere S^2_N, by interpreting $X_a^{(N)}$ as generator of a fuzzy sphere (A.1). Furthermore, we will show in section 3.1 that this type 1 vacuum (2.16) leads to spontaneous symmetry breaking, with low-energy (unbroken) gauge group $SU(n)$. The low-energy sector of the model can then be understood as compactification and dimensional reduction on this internal fuzzy sphere.

Let us discuss equation (2.17) in more detail. It can of course be solved exactly, but an expansion around $\alpha = 1$ is more illuminating. To simplify the analysis we assume

$$d = h = 0 \tag{2.18}$$

from now on, and assume furthermore that

$$a^2 \approx \frac{1}{\tilde{g}^2} \tag{2.19}$$

have the same order of magnitude. Defining the *real* number $\tilde{N}$ by

$$\tilde{b} = \frac{1}{4}(\tilde{N}^2 - 1), \tag{2.20}$$

one finds

$$\alpha = 1 - \frac{m}{N} + \frac{m(m+1)}{N^2} + O(\frac{1}{N^3}) \qquad \text{where } m = N - \tilde{N}, \tag{2.21}$$

assuming N to be large and m small. Notice that a does not enter to leading order. This can be understood by noting that the first term in (2.17) is dominating under these assumptions, which determines α to be (2.21) to leading order. The potential $V(\phi)$ is then dominated by the term

$$\frac{1}{\tilde{g}^2} F^\dagger_{ab} F_{ab} = \frac{1}{2\tilde{g}^2}\, m^2\, 1\!\!1 \; + O(\frac{1}{N}), \tag{2.22}$$

while $(\phi_a\phi_a + \tilde{b})^2 = O(\frac{1}{N^2})$. There is a deeper reason for this simple result: If $\tilde{N} \in \mathbb{N}$, then the solution (2.16) can be interpreted as a fuzzy sphere $S^2_{\tilde{N}}$ carrying a magnetic monopole of strength m, as shown explicitly in [5]; see also [22, 23]. Then (2.22) is indeed the action of the monopole field strength.

Type 2 vacuum. It is now easy to see that for suitable parameters, the vacuum will indeed consist of several distinct blocks. This will typically be the case if $\mathcal{N}$ is not divisible by the dimension of the irrep whose Casimir is closest to $\tilde{b}$.

Consider again a solution (2.15) with n_i blocks of size $N_i = \tilde{N} + m_i$, assuming that $\tilde{N}$ is large and $\frac{m_i}{\tilde{N}} \ll 1$. Generalizing (2.22), the action is then given by

$$V(\phi) = Tr\Big(\frac{1}{2\tilde{g}^2} \sum_i n_i\, m_i^2\, 1\!\!1_{N_i} \; + O(\frac{1}{N_i})\Big) \approx \frac{1}{2\tilde{g}^2}\, \frac{\mathcal{N}}{k}\, \sum_i n_i\, m_i^2 \tag{2.23}$$

where $k = \sum n_i$ is the total number of irreps, and the solution can be interpreted in terms of "instantons" (nonabelian monopoles) on the internal fuzzy sphere [5]. Hence in order to determine the solution of type (2.15) with minimal action, we simply have to minimize $\sum_i n_i\, m_i^2$, where the $m_i \in \mathbb{Z} - \tilde{N}$ satisfy the constraint $\sum n_i\, m_i = \mathcal{N} - k\tilde{N}$.

It is now easy to see that as long as the approximations used in (2.23) are valid, the vacuum is given by a partition consisting of blocks with no more than 2 distinct sizes N_1, N_2 which satisfy $N_2 = N_1 + 1$. The follows from the convexity of (2.23): assume that the vacuum is given by a configuration with 3 or more different blocks of size $N_1 < N_2 < \ldots < N_k$. Then the action (2.23) could be lowered by modifying the configuration as follows: reduce n_1 and n_k by one, and add 2 blocks of size $N_1 + 1$ and $N_k - 1$. This preserves the overall dimension, and it is easy to check (using convexity) that the action (2.23) becomes smaller. This argument can be applied as long as there are 3 or more different blocks, or 2 blocks with $|N_2 - N_1| \geq 2$. Therefore if $\mathcal{N}$ is large, the solution with minimal potential among all possible partitions (2.15) is given either by a type 1 vacuum, or takes the form

$$\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)} \otimes 1\!\!1_{n_1} & 0 \\ 0 & \alpha_2\, X_a^{(N_2)} \otimes 1\!\!1_{n_2} \end{pmatrix}, \tag{2.24}$$

where the integers N_1, N_2 satisfy

$$\mathcal{N} = N_1 n_1 + N_2 n_2, \qquad N_2 = N_1 + 1. \tag{2.25}$$

A vacuum of the form (2.24) will be denoted as "type 2 vacuum", and is the generic case. In particular, the integers n_1 and n_2 are determined dynamically. This conclusion might be altered for nonzero d, h or by a violation of the approximations used in (2.23). We will show in section 3.2 that this type of vacuum leads to a low-energy (unbroken) gauge group $SU(n_1) \times SU(n_2) \times U(1)$, and the low-energy sector can be interpreted as dimensional reduction of a higher-dimensional gauge theory on an internal fuzzy sphere, with features similar to a GUT model with SSB $SU(n_1 + n_2) \to SU(n_1) \times SU(n_2) \times U(1)$ via an adjoint Higgs. Furthermore, since the vacuum (2.24) can be interpreted as a fuzzy sphere with nontrivial magnetic flux [5], one can expect to obtain massless chiral fermions in the low-energy action. This will be worked out in detail in a forthcoming publication.

In particular, it is interesting to see that gauge groups which are products of more than two simple components (apart from U(1)) do not occur in this model.

Type 3 vacuum. Finally, it could be that the vacuum is of a type different from (2.15), e.g. with off-diagonal corrections such as

$$\phi_a = \begin{pmatrix} \alpha_1 X_a^{(N_1)} \otimes 1\!\!1_{n_1} & \varphi_a \\ -\varphi_a^\dagger & \alpha_2 X_a^{(N_2)} \otimes 1\!\!1_{n_2} \end{pmatrix} \tag{2.26}$$

for some small φ_a. We will indeed provide evidence for the existence of such a vacuum below, and argue that it leads to a further SSB. This might play a role similar to low-energy ("electroweak") symmetry breaking, which will be discussed in more detail below. In particular, it is interesting to note that the φ_a will no longer be in the adjoint of the low-energy gauge group. A possible way to obtain a SSB scenario close to the standard model is discussed in section 3.4.

2.2 Emergence of extra dimensions and the fuzzy sphere

Before discussing these vacua and the corresponding symmetry breaking in more detail, we want to explain the geometrical interpretation, assuming first that the vacuum has the form (2.16). The $X_a^{(N)}$ are then interpreted as coordinate functions (generators) of a fuzzy sphere S^2_N, and the "scalar" action

$$S_\phi = TrV(\phi) = Tr\Big(a^2(\phi_a\phi_a + \tilde{b})^2 + \frac{1}{\tilde{g}^2} F^\dagger_{ab} F_{ab}\Big) \tag{2.27}$$

for $\mathcal{N} \times \mathcal{N}$ matrices ϕ_a is precisely the action for a $\mathrm{U}(n)$ Yang-Mills theory on S^2_N with coupling $\tilde{g}$, as shown in [5] and reviewed in section B. In fact, the "unusual" term $(\phi_a\phi_a+\tilde{b})^2$ is essential for this interpretation, since it stabilizes the vacuum $\phi_a = X_a^{(N)}$ and gives a large mass to the extra "radial" scalar field which otherwise arises. The fluctuations of $\phi_a = X_a^{(N)} + A_a$ then provide the components A_a of a higher-dimensional gauge field $A_M = (A_\mu, A_a)$, and the action (2.1) can be interpreted as YM theory on the 6-dimensional space $M^4 \times S^2_N$, with gauge group depending on the particular vacuum. Note that e.g. for the type 1 vacuum, the local gauge transformations $\mathrm{U}(\mathcal{N})$ can indeed be interpreted as local $\mathrm{U}(n)$ gauge transformations on $M^4 \times S^2_N$.

In other words, the scalar degrees of freedom ϕ_a conspire to form a fuzzy space in extra dimensions. We therefore interpret the vacuum (2.16) as describing dynamically generated extra dimensions in the form of a fuzzy sphere S^2_N, with an induced Yang-Mills action on S^2_N. This geometrical interpretation will be fully justified in section 3 by working out the spectrum of Kaluza-Klein modes. The effective low-energy theory is then given by the zero modes on S^2_N, which is analogous to the models considered in [2]. However, in the present approach we have a clear dynamical selection of the geometry due to the first term in (2.27).

It is interesting to recall here the running of the coupling constants under the RG as discussed above. The logarithmic running of R implies that the scale of the internal spheres is only mildly affected by the RG flow. However, $\tilde{b}$ is running essentially quadratically, hence is generically large. This is quite welcome here: starting with some large $\mathcal{N}$, $\tilde{b} \approx C_2(\tilde{N})$ must indeed be large in order to lead to the geometric interpretation discussed above. Hence the problems of naturalness or fine-tuning appear to be rather mild here.

3. Kaluza-Klein modes, dimensional reduction, and symmetry breaking

We now study the model (2.1) in more detail. Let us emphasize again that this is a 4-dimensional renormalizable gauge theory, and there is no fuzzy sphere or any other extra-dimensional structure to start with. We have already discussed possible vacua of the potential (2.27), depending on the parameters $a, \tilde{b}, \tilde{g}$ and $\mathcal{N}$. This is a nontrivial problem, the full solution of which is beyond the scope of this paper. We restrict ourselves here to the simplest types of vacua discussed in section 2.1, and derive some of the properties of the resulting low-energy models, such as the corresponding low-energy gauge groups and the excitation spectrum. In particular, we exhibit the tower of Kaluza-Klein modes in the different cases. This turns out to be consistent with an interpretation in terms of compactification on an internal sphere, demonstrating without a doubt the emergence of fuzzy internal dimensions. In particular, the scalar fields ϕ_a become gauge fields on the fuzzy sphere.

3.1 Type 1 vacuum and $\mathrm{SU}(n)$ gauge group

Let us start with the simplest case, assuming that the vacuum has the form (2.16). We want to determine the spectrum and the representation content of the gauge field A_μ. The structure of $\phi_a = \alpha\, X_a^{(N)} \otimes \mathbb{1}_n$ suggests to consider the subgroups $\mathrm{SU}(N)\times\mathrm{SU}(n)$ of $\mathrm{SU}(\mathcal{N})$, where

$$K := \mathrm{SU}(n) \tag{3.1}$$

is the commutant of ϕ_a i.e. the maximal subgroup of $\mathrm{SU}(\mathcal{N})$ which commutes with all ϕ_a, $a = 1, 2, 3$; this follows from Schur's Lemma. K will turn out to be the effective (low-energy) unbroken 4-dimensional gauge group.

We could now proceed in a standard way arguing that $\mathrm{SU}(\mathcal{N})$ is spontaneously broken to K since ϕ_a takes a VEV as in (2.16), and elaborate the Higgs mechanism. This is essentially what will be done below, however in a language which is very close to the picture of compactification and KK modes on a sphere in extra dimensions. This is appropriate here, and leads to a description of the low-energy physics of this model as a dimensionally reduced $\mathrm{SU}(n)$ gauge theory.

Kaluza-Klein expansion on S^2_N. Interpreting the $X_a^{(N)}$ as generators of the fuzzy sphere S^2_N, we can decompose the full 4-dimensional $\mathfrak{su}(\mathcal{N})$-valued gauge fields A_μ into spherical harmonics $Y^{lm}(x)$ on the fuzzy sphere S^2_N with coordinates x_a:

$$A_\mu = \sum_{0\le l\le N, |m|\le l} Y^{lm}(x) \otimes A_{\mu,lm}(y) = A_\mu(x,y). \tag{3.2}$$

The Y^{lm} are by definition irreps under the $\mathrm{SU}(2)$ rotations on S^2_N, and form a basis of Hermitian $N \times N$ matrices; for more details see section A. The $A_{\mu,lm}(y)$ turn out to be $\mathfrak{u}(n)$-valued gauge and vector fields on M^4. Using this expansion, we can interpret $A_\mu(x,y)$ as $\mathfrak{u}(n)$-valued functions on $M^4 \times S^2_N$, expanded into the Kaluza-Klein modes (i.e. harmonics) of S^2_N.

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The scalar fields ϕ_a with potential (2.27) and vacuum (2.16) should be interpreted as "covariant coordinates" on S^2_N which describe U(n) Yang-Mills theory on S^2_N. This means that the fluctuations A_a of these covariant coordinates

$$\phi_a = \alpha\, X_a^{(N)} \otimes \mathbb{1}_n + A_a \tag{3.3}$$

should be interpreted as gauge fields on the fuzzy sphere, see (B.4). They can be expanded similarly as

$$A_a = \sum_{l,m} Y^{lm}(x) \otimes A_{a,lm}(y) = A_a(x,y), \tag{3.4}$$

interpreted as functions (or 1-form) on $M^4 \times S^2_N$ taking values in $\mathfrak{u}(n)$. One can then interpret $A_M(x,y) = (A_\mu(x,y), A_a(x,y))$ as $\mathfrak{u}(n)$-valued gauge or vector fields on $M^4 \times S^2_N$.

Given this expansion into KK modes, we will show that only $A_{\mu,00}(y)$ (i.e. the dimensionally reduced gauge field) becomes a massless $\mathfrak{su}(n)$-valued[2] gauge field in 4D, while all other modes $A_{\mu,lm}(y)$ with $l \geq 1$ constitute a tower of Kaluza-Klein modes with large mass gap, and decouple for low energies. The existence of these KK modes firmly establishes our claim that the model develops dynamically extra dimensions in the form of S^2_N. This geometric interpretation is hence forced upon us, provided the vacuum has the form (2.16). The scalar fields $A_a(x,y)$ will be analyzed in a similar way below, and provide no additional massless degrees of freedom in 4 dimensions. More complicated vacua will have a similar interpretation. Remarkably, our model is fully renormalizable in spite of its higher-dimensional character, in contrast to the commutative case; see also [3].

Computation of the KK masses. To justify these claims, let us compute the masses of the KK modes (3.2). They are induced by the covariant derivatives $\int Tr(D_\mu\phi_a)^2$ in (2.1),

$$\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + 2(\partial_\mu\phi_a^\dagger)[A_\mu,\phi_a] + [A_\mu,\phi_a]^\dagger[A_\mu,\phi_a]). \tag{3.5}$$

The most general scalar field configuration can be written as

$$\phi_a(y) = \alpha(y) X_a^{(N)} \otimes \mathbb{1}_n + A_a(x,y) \tag{3.6}$$

where $A_a(x,y)$ is interpreted as gauge field on the fuzzy sphere S^2_N for each $y \in M^4$. We allow here for a y–dependent $\alpha(y)$ (which could have been absorbed in $A_a(x,y)$), because it is naturally interpreted as the Higgs field responsible for the symmetry breaking SU($\mathcal{N}$) $\to$ SU(n). As usual, the last term in (3.5) leads to the mass terms for the gauge fields A_μ in the vacuum $\phi_a(y) = \alpha X_a^{(N)} \otimes \mathbb{1}_n$, provided the mixed term which is linear in A_μ vanishes in a suitable gauge. This is usually achieved by going to the unitary gauge. In the present case this is complicated by the fact that we have 3 scalars in the adjoint, and there is no obvious definition of the unitary gauge; in fact, there are are too many scalar degrees of freedom as to gauge away that term completely. However, we can choose a gauge where

[2] note that $A_{\mu,00}(y)$ is traceless, while $A_{\mu,lm}(y)$ is not in general

all quadratic contributions of that term vanish, leaving only cubic interaction terms. To see this, we insert (3.6) into the term $(\partial_\mu \phi_a^\dagger)[A_\mu, \phi_a]$ in (3.5), which gives

$$\int Tr A_\mu [\phi_a, \partial_\mu \phi_a^\dagger] =$$
$$\int Tr A_\mu \Big(\alpha [X_a, \partial_\mu A_a(x,y)] + [A_a(x,y), \partial_\mu \alpha\, X_a] + [A_a(x,y), \partial_\mu A_a(x,y)] \Big).$$

Now we partially fix the gauge by imposing the "internal" Lorentz gauge $[X_a, A_a] = 0$ at each point y. This is always possible[3], and the above simplifies as

$$\int Tr A_\mu [\phi_a, \partial_\mu \phi_a^\dagger] = \int Tr A_\mu [A_a(x,y), \partial_\mu A_a(x,y)] =: S_{int}. \tag{3.7}$$

This contains only cubic interaction terms, which are irrelevant for the computation of the masses. We can therefore proceed by setting $\phi_a(y) = \alpha X_a^{(N)} \otimes 1\!\!1_n$ and inserting the expansion (3.2) of A_μ into the last term of (3.5). Noting that $i[X_a, A_\mu] = J_a A_\mu = \sum_{l,m} A_{\mu,lm}(y)\, J_a Y^{lm}$ is simply the action of SU(2) on the fuzzy sphere, it follows that $Tr[X_a, A_\mu][X_a, A_\mu]$ is the quadratic Casimir on the modes of A_μ which are orthogonal, and we obtain

$$\int Tr (D_\mu \phi_a)^\dagger D_\mu \phi_a = \int Tr(\partial_\mu \phi_a^\dagger \partial_\mu \phi_a + \sum_{l,m} \alpha^2\, l(l+1)\, A_{\mu,lm}(y)^\dagger A_{\mu,lm}(y)) + S_{int}. \tag{3.8}$$

Therefore the 4-dimensional $\mathfrak{u}(n)$ gauge fields $A_{\mu,lm}(y)$ acquire a mass

$$m_l^2 = \frac{\alpha^2 g^2}{R^2}\, l(l+1) \tag{3.9}$$

reinserting the parameter R (2.6) which has dimension length. This is as expected for higher KK modes, and determines the radius of the internal S^2 to be

$$r_{S^2} = \frac{\alpha}{g} R \tag{3.10}$$

where $\alpha \approx 1$ according to (2.21). In particular, only $A_\mu(y) \equiv A_{\mu,00}(y)$ survives as a massless 4-dimensional $\mathfrak{su}(n)$ gauge field. The low-energy effective action for the gauge sector is then given by

$$S_{LEA} = \int d^4 y\, \frac{1}{4g^2}\, Tr_n\, F_{\mu\nu}^\dagger F_{\mu\nu}, \tag{3.11}$$

where $F_{\mu\nu}$ is the field strength of the low-energy $\mathfrak{su}(n)$ gauge fields, dropping all other KK modes whose mass scale is set by $\frac{1}{R}$. For $n = 1$, there is no massless gauge field. However we would find a massless U(1) gauge field if we start with a $U(\mathcal{N})$ gauge theory rather than $SU(\mathcal{N})$.

[3] even though this gauge is commonly used in the literature on the fuzzy sphere, a proof of existence has apparently not been given. It can be proved by extremizing the real function $Tr(X_a \phi_a)$ on a given gauge orbit, which is compact; the e.o.m. then implies $[X_a, \phi_a] = 0$.

Scalar sector. We now expand the most general scalar fields ϕ_a into modes, singling out the coefficient of the "radial mode" as

$$\phi_a(y) = X_a^{(N)} \otimes (\alpha 1\!\!1_n + \varphi(y)) + \sum_k A_{a,k}(x) \otimes \varphi_k(y). \tag{3.12}$$

Here $A_{a,k}(x)$ stands for a suitable basis labeled by k of fluctuation modes of gauge fields on S_N^2, and $\varphi(y)$ resp. $\varphi_k(y)$ are $\mathfrak{u}(n)$-valued. We expect that all fluctuation modes in the expansion (3.12) have a large mass gap of the order of the KK scale, which is indeed the case as shown in detail in section C. Therefore we can drop all these modes for the low-energy sector. However, the field $\varphi(y)$ plays a somewhat special role. It corresponds to fluctuations of the radius of the internal fuzzy sphere, which is the order parameter responsible for the SSB $SU(\mathcal{N}) \to SU(n)$, and assumes the value $\alpha 1\!\!1_n$ in (3.12). $\varphi(y)$ is therefore the Higgs which acquires a positive mass term in the broken phase, which can be obtained by inserting $\phi_a(y) = X_a^{(N)} \otimes (\alpha 1\!\!1_n + \varphi(y))$ into $V(\phi)$. This mass is dominated by the first term in (2.7) (assuming $a^2 \approx \frac{1}{\tilde{g}^2}$), of order

$$V(\varphi(y)) \approx N\left(a^2 C_2(N)^2 \varphi(y)^2 + O(\varphi^3)\right) \tag{3.13}$$

for large $\mathcal{N}$ and N. The full potential for φ is of course quartic.

We conclude that our model indeed behaves like a $U(n)$ gauge theory on $M^4 \times S_N^2$, with the expected tower of KK modes on the fuzzy sphere S_N^2 of radius (3.10). The low-energy effective action is given by the lowest KK mode, which is

$$S_{LEA} = \int d^4y\, Tr_n \left(\frac{1}{4g^2} F_{\mu\nu}^\dagger F_{\mu\nu} + D_\mu\varphi(y) D_\mu\varphi(y)\, N C_2(N) + N a^2 C_2(N)^2 \varphi(y)^2\right) + S_{int} \tag{3.14}$$

for the $SU(n)$ gauge field $A_\mu(y) \equiv A_{\mu,00}(y)$. In (3.14) we also keep the Higgs field $\varphi(y)$, even though it acquires a large mass

$$m_\varphi^2 = \frac{a^2}{R^2} C_2(N) \tag{3.15}$$

reinserting R.

3.2 Type 2 vacuum and $SU(n_1) \times SU(n_2) \times U(1)$ gauge group

For different parameters in the potential, we can obtain a different vacuum, with different low-energy gauge group. Assume now that the vacuum has the form (2.24). The structure of ϕ_a suggests to consider the subgroups $(SU(N_1) \times SU(n_1)) \times (SU(N_2) \times SU(n_2)) \times U(1)$ of $SU(\mathcal{N})$, where

$$K := SU(n_1) \times SU(n_2) \times U(1) \tag{3.16}$$

is the maximal subgroup of $SU(\mathcal{N})$ which commutes with all ϕ_a, $a = 1,2,3$ (this follows from Schur's Lemma). Here the $U(1)$ factor is embedded as

$$\mathfrak{u}(1) \sim \begin{pmatrix} \frac{1}{N_1 n_1} 1\!\!1_{N_1 \times n_1} & \\ & -\frac{1}{N_2 n_2} 1\!\!1_{N_2 \times n_2} \end{pmatrix} \tag{3.17}$$

which is traceless. K will again be the effective (low-energy) 4-dimensional gauge group.

We now repeat the above analysis of the KK modes and their effective 4-dimensional mass. First, we write

$$A_\mu = \begin{pmatrix} A_\mu^1 & A_\mu^+ \\ A_\mu^- & A_\mu^2 \end{pmatrix} \tag{3.18}$$

according to (2.24), where $(A_\mu^+)^\dagger = -A_\mu^-$. The masses of the gauge bosons are again induced by the last term in (3.5). Consider the term $[\phi_a, A_\mu] = [\alpha_1 X_a^{(N_1)} + \alpha_2 X_a^{(N_2)}, A_\mu]$. For the diagonal fluctuations $A_\mu^{1,2}$, this is simply the adjoint action of $X_a^{(N_1)}$. For the off-diagonal modes $A_\mu^\pm$, we can get some insight by assuming first $\alpha_1 = \alpha_2$. Then the above commutator is $X^{(N_1)} A_\mu^+ - A_\mu^+ X^{(N_2)}$, reflecting the representation content $A_\mu^+ \in (N_1)\otimes(N_2)$ and $A_\mu^- \in (N_2)\otimes(N_1)$. Assuming $N_1 - N_2 = k > 0$, this implies in particular that there are *no zero modes for the off-diagonal blocks*, rather the lowest angular momentum is k. They can be interpreted as being sections on a monopole bundle with charge k on $S^2_{N_1}$, cf. [5]. The case $\alpha_1 \neq \alpha_2$ requires a more careful analysis as indicated below. In any case, we can again expand A_μ into harmonics,

$$A_\mu = \sum_{l,m} \begin{pmatrix} Y^{lm(N_1)}\, A^1_{\mu,lm}(y) & Y^{lm(+)}\, A^+_{\mu,lm}(y) \\ Y^{lm(-)}\, A^-_{\mu,lm}(y) & Y^{lm(N_2)}\, A^2_{\mu,lm}(y) \end{pmatrix} = A_\mu(x,y) \tag{3.19}$$

setting $Y^{lm(N)} = 0$ if $l > 2N$. Then the $A^{1,2}_{\mu,lm}(y)$ are $\mathfrak{u}(n_1)$ resp. $\mathfrak{u}(n_2)$-valued gauge resp. vector fields on M^4, while $A^\pm_{\mu,lm}(y)$ are vector fields on M^4 which transform in the bifundamental $(n_1,\overline{n}_2)$ resp. $(n_2,\overline{n}_1)$ of $\mathfrak{u}(n_1)\times\mathfrak{u}(n_2)$.

Now we can compute the masses of these fields. For the diagonal blocks this is the same as in section 3.1, while the off-diagonal components can be handled by writing

$$Tr([\phi_a, A_\mu][\phi_a, A_\mu]) = 2Tr(\phi_a A_\mu \phi_a A_\mu - \phi_a \phi_a A_\mu A_\mu). \tag{3.20}$$

This gives

$$\begin{aligned}\int Tr(D_\mu\phi_a)^\dagger D_\mu\phi_a = \int Tr\Big(\partial_\mu\phi_a^\dagger\partial_\mu\phi_a + \sum_{l\geq 0}(m^2_{l,1}\, A^{1\dagger}_{\mu,lm}(y) A^1_{\mu,lm}(y) + m^2_{l,2}\, A^{2\dagger}_{\mu,lm}(y) A^2_{\mu,lm}(y)) \\ + \sum_{l\geq k} 2m^2_{l;\pm}(A^+_{\mu,lm}(y))^\dagger A^+_{\mu,lm}(y)\Big)\end{aligned} \tag{3.21}$$

similar as in (3.8), with the same gauge choice and omitting cubic interaction terms. In particular, the diagonal modes acquire a KK mass

$$m^2_{l,i} = \frac{\alpha_i^2 g^2}{R^2}\, l(l+1) \tag{3.22}$$

completely analogous to (3.9), while the off-diagonal modes acquire a mass

$$\begin{aligned} m^2_{l;\pm} &= \frac{g^2}{R^2}\,\big(\alpha_1\alpha_2\, l(l+1) + (\alpha_1-\alpha_2)(X_2^2\alpha_2 - X_1^2\alpha_1)\big) \\ &\approx \frac{g^2}{R^2}\,\Big(l(l+1) + \frac{1}{4}(m_2-m_1)^2 \;+ O(\frac{1}{\mathcal{N}})\Big)\end{aligned} \tag{3.23}$$

using (2.21) for $\alpha_i \approx 1$. In particular, all masses are positive.

We conclude that the gauge fields $A^{1,2}_{\mu,lm}(y)$ have massless components $A^{1,2}_{\mu,00}(y)$ which take values in $\mathfrak{su}(n_i)$ due to the KK-mode $l = 0$ (as long as $n_i > 1$), while the bifundamental fields $A^{\pm}_{\mu,lm}(y)$ have no massless components. Note that the mass scales of the diagonal modes (3.22) and the off-diagonal modes (3.23) are essentially the same. This result is similar to the breaking $SU(n_1+n_2) \to SU(n_1) \times SU(n_2) \times U(1)$ through an adjoint Higgs, such as in the $SU(5) \to SU(3) \times SU(2) \times U(1)$ GUT model. In that case, one also obtains massive ("ultraheavy") gauge fields in the bifundamental, whose mass should therefore be identified in our scenario with the mass (3.23) of the off-diagonal massive KK modes $A^{\pm}_{\mu,lm}(y)$. The U(1) factor (3.17) corresponds to the massless components $A^{1,2}_{\mu,00}(y)$ above, which is now present even if $n_i = 1$. We therefore found results comparable to [24], but within the framework of a renormalizable theory.

The appropriate interpretation of this vacuum is as a gauge theory on $M^4 \times S^2$, compactified on S^2 which carries a magnetic flux with monopole number $|N_1 - N_2|$. This leads to a low-energy action with gauge group $SU(n_1) \times SU(n_2) \times U(1)$. The existence of a magnetic flux is particularly interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions. This issue will be studied in detail elsewhere.

Repeating the analysis of fluctuations for the scalar fields is somewhat messy, and will not be given here. However since the vacuum (2.24) is assumed to be stable, all fluctuations in the ϕ_a will again be massive with mass presumably given by the KK scale, and can therefore be omitted for the low-energy theory. Again, one could interpret the fluctuations $\varphi_{1,2}(y)$ of the radial modes $X^{(N_{1,2})}_a \otimes (\alpha_{1,2} + \varphi_{1,2}(y))$ as low-energy Higgs in analogy to (3.12), responsible for the symmetry breaking $SU(n_1+n_2) \to SU(n_1) \times SU(n_2) \times U(1)$.

3.3 Type 3 vacuum and further symmetry breaking

Finally consider a vacuum of the form (2.26). The additional fields φ_a transform in the bifundamental of $SU(n_1) \times SU(n_2)$ and lead to further SSB. Of particular interest is the simplest case

$$\phi_a = \begin{pmatrix} \alpha_1\, X^{(N_1)}_a \otimes \mathbb{1}_n & \varphi_a \\ -\varphi^\dagger_a & \alpha_2\, X^{(N_2)}_a \end{pmatrix} \tag{3.24}$$

corresponding to a would-be gauge group $SU(n) \times U(1)$ according to section 3.2, which will be broken further. Then $\varphi_a = \begin{pmatrix} \varphi_{a,1} \\ \vdots \\ \varphi_{a,n} \end{pmatrix}$ lives in the fundamental of $SU(n)$ charged under U(1), and transforms as $(N_1) \otimes (N_2)$ under the SO(3) corresponding to the fuzzy sphere(s). As discussed below, by adding a further block, one can get somewhat close to the standard model, with φ_a being a candidate for a low-energy Higgs.

We will argue that there is indeed such a solution of the equation of motion (2.14) for $|N_1 - N_2| = 2$. Note that since $\varphi_a \in (N_1) \otimes (N_2) = (|N_1 - N_2| + 1) \oplus \ldots \oplus (N_1 + N_2 - 1)$, it can transform as a vector under SO(3) only in that case. Hence assume $N_1 = N_2 + 2$, and define $\varphi_a \in (N_1) \otimes (N_2)$ to be the unique component which transform as a vector in the adjoint. One can then show that

$$\phi_a\phi_a = -\begin{pmatrix} \alpha_1^2\, C_2(N_1)\otimes \mathbb{1}_{n_1} - \frac{h}{N_1} & 0 \\ 0 & \alpha_2^2\, C_2(N_2) - \frac{h}{N_2} \end{pmatrix} \tag{3.25}$$

where h is a normalization constant, and

$$\varepsilon_{abc}\phi_b\phi_c = \begin{pmatrix} (\alpha_1^2 - \frac{g_1}{N_1}\frac{h}{C_2(N_1)})\, X_a^{(N_1)} & (\alpha_1 g_1 + \alpha_2 g_2)\varphi_a \\ -(\alpha_1 g_1 + \alpha_2 g_2)\varphi_a^\dagger & (\alpha_2^2 - \frac{g_2}{N_2}\frac{h}{C_2(N_2)})\, X_a^{(N_2)} \end{pmatrix} \tag{3.26}$$

where $g_1 = \frac{N_1+1}{2}$, $g_2 = -\frac{N_2-1}{2}$. This has the same form as (3.24) but with different parameters. We now have 3 parameters α_1, α_2, h at our disposal, hence generically this Ansatz will provide solutions of the e.o.m. (2.14) which amounts to 3 equations for the independent blocks. It remains to be seen whether they are energetically favorable. This will be studied in a future publication.

The commutant K and further symmetry breaking. To determine the low-energy gauge group i.e. the maximal subgroup K commuting with the solution ϕ_a of type (3.24), consider

$$\begin{aligned} &\varepsilon_{abc}\phi_b\phi_c - (\alpha_1 g_1 + \alpha_2 g_2)\phi_a = \\ &\begin{pmatrix} (\alpha_1^2 - \alpha_1(\alpha_1 g_1 + \alpha_2 g_2) - \frac{g_1}{N_1}\frac{h}{C_2(N_1)})\, X_a^{(N_1)} & 0 \\ 0 & (\alpha_2^2 - \alpha_2(\alpha_1 g_1 + \alpha_2 g_2) - \frac{g_2}{N_2}\frac{h}{C_2(N_2)})\, X_a^{(N_2)} \end{pmatrix} \end{aligned} \tag{3.27}$$

Unless one of the two coefficients vanishes, this implies that K must commute with (3.27), hence $K = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$ is a subgroup of $\mathrm{SU}(n_1)\times\mathrm{SU}(n_2)\times\mathrm{U}(1)$; here we focus on $\mathrm{SU}(n_2) = \mathrm{SU}(1)$ being trivial. Then (3.24) implies that $k_1\varphi_a = \varphi_a k_2$ for $k_i \in K_i$, which means that φ_a is an eigenvector of k_1 with eigenvalue k_2. Using a $\mathrm{SU}(n_1)$ rotation, we can assume that $\varphi_a^T = (\varphi_{a,1}, 0, \ldots, 0)$. Taking into account the requirement that K is traceless, it follows that $K \cong K_1 \cong \mathrm{SU}(n_1-1) \subset \mathrm{SU}(n_1)$. Therefore the gauge symmetry is broken to $\mathrm{SU}(n_1-1)$. This can be modified by adding a further block as discussed below.

3.4 Towards the standard model

Generalizing the above considerations, we can construct a vacuum which is quite close to the standard model. Consider

$$\mathcal{N} = N_1 n_1 + N_2 n_2 + N_3, \tag{3.28}$$

for $n_1 = 3$ and $n_2 = 2$. As discussed above, we expect a vacuum of the form

$$\phi_a = \begin{pmatrix} \alpha_1\, X_a^{(N_1)}\otimes \mathbb{1}_3 & 0 & 0 \\ 0 & \alpha_2\, X_a^{(N_2)}\otimes \mathbb{1}_2 & \varphi_a \\ 0 & -\varphi_a^\dagger & \alpha_3\, X_a^{(N_3)} \end{pmatrix} \tag{3.29}$$

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if $\tilde{b} \approx C_2(N_1)$ and $N_1 \approx N_2 = N_3 \pm 2$. Then the unbroken low-energy gauge group would be

$$K = \mathrm{SU}(3) \times \mathrm{U}(1)_Q \times \mathrm{U}(1)_F, \tag{3.30}$$

with $\mathrm{U}(1)_F$ generated by the traceless generator

$$u(1)_F \sim \begin{pmatrix} \frac{1}{3N_1}\,\mathbb{1}_{3N_1} & \\ & -\frac{1}{D}\,\mathbb{1}_D \end{pmatrix} \tag{3.31}$$

where $D = 2N_2 + N_3$, and $\mathrm{U}(1)_Q$ generated by the traceless generator

$$u(1)_Q \sim \begin{pmatrix} \frac{1}{3N_1}\,\mathbb{1}_{3N_1} & & \\ & -\frac{1}{N_2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\mathbb{1}_{N_2} & \\ & & 0 \end{pmatrix}. \tag{3.32}$$

assuming that $\varphi_a^T = (\varphi_{a,1}, 0)$. This is starting to be reminiscent of the standard model, and will be studied in greater detail elsewhere. However, we should recall that the existence of a *vacuum* of this form has not been established at this point.

Relation with CSDR scheme

Let us compare the results of this paper with the CSDR construction in [2]. In that paper, effective 4-dimensional models are constructed starting from gauge theory on $M^4 \times S^2_{\mathcal{N}}$, by imposing CSDR constraints following the general ideas of [25–28]. These constraints boiled down to choosing embeddings ω_a, $a = 1, 2, 3$ of $\mathrm{SU}(2) \subset \mathrm{SU}(\mathcal{N})$, which determine the unbroken gauge field as the commutant of ω_a, and the low-energy (unbroken) Higgs by $\varphi_a \sim \omega_a$. This is similar to the "choice" of vacuum in the present paper, such as (2.16), (2.24), identifying ω_a with $\oplus_i X_a^{N_i}$ as in (2.15). The solutions of these constraints can be formally identified with the zero modes $A_{\mu,00}$ of the KK-tower of gauge fields (3.2), resp. the vacuum of the scalar sector (3.12). In this sense, the possible vacua (2.15) could be interpreted as solutions of the CSDR constraints in [2] on a given fuzzy sphere.

However, there are important differences. First, the present approach provides a clear dynamical mechanism which chooses a unique vacuum. This depends crucially on the first term in (2.7), that removes the degeneracy of all possible embeddings of SU(2), which have vanishing field strength F_{ab}. Moreover, it may provide an additional mechanism for further symmetry breaking as discussed in section 3.3. Another difference is that the starting point in [2] is a 6-dimensional gauge theory with some given gauge group, such as U(1). This is not the case in present paper, where the 6-dimensional gauge group depends on the parameters of the model.

4. Discussion

We have presented a renormalizable 4-dimensional $\mathrm{SU}(\mathcal{N})$ gauge theory with a suitable multiplet of scalars, which dynamically develops fuzzy extra dimensions that form a fuzzy sphere. The model can then be interpreted as 6-dimensional gauge theory, with gauge group and geometry depending on the parameters in the original Lagrangian. We explicitly find the tower of massive Kaluza-Klein modes, consistent with an interpretation as compactified higher-dimensional gauge theory, and determine the effective compactified gauge theory. Depending on the parameters of the model the low-energy gauge group can be $SU(n)$, or broken further e.g. to $SU(n_1) \times SU(n_2) \times U(1)$, with mass scale determined by the extra dimension.

There are many remarkable aspects of this model. First, it provides an extremely simple and geometrical mechanism of dynamically generating extra dimensions, without relying on subtle dynamics such as fermion condensation and particular Moose- or Quiver-type arrays of gauge groups and couplings, such as in [1] and following work. Rather, our model is based on a basic lesson from noncommutative gauge theory, namely that noncommutative or fuzzy spaces can be obtained as solutions of matrix models. The mechanism is quite generic, and does not require fine-tuning or supersymmetry. This provides in particular a realization of the basic ideas of compactification and dimensional reduction within the framework of renormalizable quantum field theory. Moreover, we are essentially considering a large $\mathcal{N}$ gauge theory, which should allow to apply the analytical techniques developed in this context.

One of the main features of our mechanism is that the effective properties of the model including its geometry depend on the particular parameters of the Lagrangian, which are subject to renormalization. In particular, the RG flow of these parameters depends on the specific vacuum i.e. geometry, which in turn will depend on the energy scale. For example, it could be that the model assumes a "type 3" vacuum as discussed in section 3.3 at low energies, which might be quite close to the standard model. At higher energies, the parameter $\tilde{b}$ (which determines the effective gauge group and which is expected to run quadratically under the RG flow) will change, implying a very different vacuum with different gauge group etc. This suggests a rich and complicated dynamical hierarchy of symmetry breaking, which remains to be elaborated.

In particular, we have shown that the low-energy gauge group is given by $SU(n_1) \times SU(n_2) \times U(1)$ or $SU(n)$, while gauge groups which are products of more than two simple components (apart from $U(1)$) do not seem to occur in this model. The values of n_1 and n_2 are determined dynamically. Moreover, the existence of a magnetic flux in the vacua with non-simple gauge group is very interesting in the context of fermions, since internal fluxes naturally lead to chiral massless fermions. This will be studied in detail elsewhere.

There is also an intriguing analogy between our toy model and string theory, in the sense that as long as $a = 0$, there are a large number of possible vacua (given by all possible partitions (2.15)) corresponding to compactifications, with no dynamical selection mechanism to choose one from the other. Remarkably this analog of the "string vacuum problem" is simply solved by adding a term to the action.

Finally we should point out some potential problems or shortcomings of our model. First, we have not yet fully established the existence of the most interesting vacuum structure of type 3 such as in (3.24) or (3.29). This will be studied in a future paper. Even a full analysis of the fluctuations and KK modes in the scalar sector for vacuum of type

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2 has not been done, but we expect no surprises here; a numerical study is currently in progress. Finally, the use of scalar Higgs fields ϕ_a without supersymmetry may seem somewhat problematic due to the strong renormalization behavior of scalar fields. This is in some sense consistent with the interpretation as higher-dimensional gauge theory, which would be non-renormalizable in the classical case. Moreover, a large value of the quadratically divergent term $\tilde{b}$ is quite desirable here as explained in section 2.2, and does not require particular fine-tuning.

Acknowledgments

We are grateful for discussions with C. Bachas, H. Grosse and B. Jurco. This work was partially supported by the EPEAEK programme "Pythagoras" and co-founded by the European Union (75%) and the Hellenic state (25%). The work of H.S. is supported by the FWF under project P18657, and the work of P.A. is partially supported by the EC contract MRTN-CT-2004-005104 and the italian MIUR contract PRIN-2005023102. H.S. in particular acknowledges an invitation to the Technical University of Athens and to the Universita del Piemonte Orientale, Alessandria. H.S., P.A. and G.Z. also acknowledge an invitation to the Werner-Heisenberg-Institut Munich where some of this work has been carried out.

A. The fuzzy sphere

The fuzzy sphere [29] is a matrix approximation of the usual sphere S^2. The algebra of functions on S^2 (which is spanned by the spherical harmonics) is truncated at a given frequency and thus becomes finite dimensional. The algebra then becomes that of $N \times N$ matrices. More precisely, the algebra of functions on the ordinary sphere can be generated by the coordinates of $\mathbb{R}^3$ modulo the relation $\sum_{a=1}^{3} x_a x_a = r^2$. The fuzzy sphere S^2_N is the non-commutative manifold whose coordinate functions

$$x_a = r \frac{i}{\sqrt{C_2(N)}} X_a, \qquad x_a^\dagger = x_a \tag{A.1}$$

are $N \times N$ hermitian matrices proportional to the generators of the N-dimensional representation of SU(2). They satisfy the condition $\sum_{a=1}^{3} x_a x_a = r^2$ and the commutation relations

$$[X_a, X_b] = \varepsilon_{abc} X_c \ . \tag{A.2}$$

For $N \to \infty$, one recovers the usual commutative sphere. The best way to see this is to decompose the space of functions on S^2_N into irreps under the SU(2) rotations,

$$\begin{aligned} S^2_N \cong (N) \otimes (N) &= (1) \oplus (3) \oplus \ldots \oplus (2N-1) \\ &= \{Y^{0,0}\} \oplus \ldots \oplus \{Y^{(N-1),m}\}. \end{aligned} \tag{A.3}$$

This provides at the same time the definition of the fuzzy spherical harmonics Y^{lm}, which we normalize as

$$Tr_N \left((Y^{lm})^\dagger Y^{l'm'} \right) = \delta^{ll'} \delta^{mm'}. \tag{A.4}$$

Furthermore, there is a natural SU(2) covariant differential calculus on the fuzzy sphere. This calculus is three-dimensional, and the derivations of a function f along X_a are given by $e_a(f) = [X_a, f]$. These are essentially the angular momentum operators

$$J_a f = i e_a f = [iX_a, f], \tag{A.5}$$

which satisfy the SU(2) Lie algebra relation

$$[J_a, J_b] = i\varepsilon_{abc} J_c. \tag{A.6}$$

In the $N \to \infty$ limit the derivations e_a become $e_a = \varepsilon_{abc} x_b \partial_c$, and only in this commutative limit the tangent space becomes two-dimensional. The exterior derivative is given by

$$df = [X_a, f]\theta^a \tag{A.7}$$

where θ^a are defined to be the one-forms dual to the vector fields e_a, $< e_a, \theta^b >= \delta_a^b$. The space of one-forms is generated by the θ^a's in the sense that any one-form can be written as $\omega = \sum_{a=1}^3 \omega_a \theta^a$. The differential geometry on the product space Minkowski times fuzzy sphere, $M^4 \times S_N^2$, is easily obtained from that on M^4 and on S_N^2. For example a one-form A defined on $M^4 \times S_N^2$ is written as

$$A = A_\mu dy^\mu + A_a \theta^a \tag{A.8}$$

with $A_\mu = A_\mu(y^\mu, x_a)$ and $A_a = A_a(y^\mu, x_a)$.

For further developments see e.g. [30 − 32] and references therein.

B. Gauge theory on the fuzzy sphere

Here we briefly review the construction of YM gauge theory on S_N^2 as multi-matrix model [5, 21, 20]. Consider the action

$$S = \frac{4\pi}{\mathcal{N}}\, Tr\Big(a^2(\phi_a\phi_a + C_2(N))^2 + \frac{1}{\tilde{g}^2}\, F_{ab}^\dagger F_{ab}\Big) \tag{B.1}$$

where $\phi_a = -\phi_a^\dagger$ is an antihermitian $\mathcal{N} \times \mathcal{N}$ matrix, and define[4]

$$F_{ab} = [\phi_a, \phi_b] - \varepsilon_{abc}\phi_c\,. \tag{B.2}$$

This action is invariant under the U($\mathcal{N}$) "gauge" symmetry acting as

$$\phi_a \to U^{-1}\phi_a U.$$

A priori, we do not assume any underlying geometry, which arises dynamically. We claim that it describes U(n) YM gauge theory on the fuzzy sphere S_N^2, assuming that $\mathcal{N} = Nn$.

To see this, we first note that the action is positive definite, with global minimum $S = 0$ for the "vacuum" solution

$$\phi_a = X_a^{(N)} \otimes \mathbb{1}_n \tag{B.3}$$

[4] This can indeed be seen as components of the two-form $F = dA + AA$

where $X_a \equiv X_a^{(N)}$ are the generators of the N- dimensional irrep of SU(2). This is a first indication that the model "dynamically generates" its own geometry, which is the fuzzy sphere S^2_N. In any case, it is natural to write a general field ϕ_a in the form

$$\phi_a = X_a + A_a, \tag{B.4}$$

and to consider $A_a = \sum_\alpha A_{a,\alpha}(x)\, T_\alpha$ as functions $A_{a,\alpha}(x) = -A_{a,\alpha}(x)^\dagger$ on the fuzzy sphere S^2_N, taking value in $u(n)$ with generators T_α. The gauge transformation then takes the form

$$\begin{aligned} A_a &\to U^{-1} A_a U + U^{-1}[X_a, U] \\ &= U^{-1} A_a U - iU^{-1} J_a U, \end{aligned} \tag{B.5}$$

which is the transformation rule of a U(n) gauge field. The field strength becomes

$$\begin{aligned} F_{ab} &= [X_a, A_b] - [X_b, A_a] + [A_a, A_b] - \varepsilon_{abc} A_c \\ &= -iJ_a A_b + iJ_b A_a + [A_a, A_b] - \varepsilon_{abc} A_c. \end{aligned} \tag{B.6}$$

This look like the field strength of a nonabelian U(n) gauge field, with the caveat that we seem to have 3 degrees of freedom rather than 2. To solve this puzzle, consider again the action, writing it in the form

$$S = \frac{4\pi}{\mathcal{N}}\, Tr\Big(a^2 \varphi^2 + \frac{1}{\tilde{g}^2}\, F_{ab}^\dagger F_{ab}\Big), \tag{B.7}$$

where we introduce the scalar field

$$\varphi := \phi_a \phi_a + C_2(N) = X_a A_a + A_a X_a + A_a A_a. \tag{B.8}$$

Since only configurations where φ and F_{ab} are small will significantly contribute to the action, it follows that

$$x_a A_a + A_a x_a = O(\frac{\varphi}{N}) \tag{B.9}$$

is small. This means that A_a is tangential in the (commutative) large N limit, and 2 tangential gauge degrees of freedom[5] survive. Equivalently, one can use the scalar field $\phi = N\varphi$, which would acquire a mass of order N and decouple from the theory.

We have thus established that the matrix model (B.1) is indeed a fuzzy version of pure U(n) YM theory on the sphere, in the sense that it reduces to the commutative model in the large N limit. Without the term $(\phi_a\phi_a + C_2(N))^2$, the scalar field corresponding to the radial component of A_a no longer decouples and leads to a different model.

The main message to be remembered is the fact that the matrix model (B.1) without any further geometrical assumptions dynamically generates the space S^2_N, and the fluctuations turn out to be gauge fields governed by a U(n) YM action. Furthermore, the vacuum has no flat directions[6], as we demonstrate explicitly in the following section.

[5] to recover the familiar form of gauge theory, one needs to rotate the components locally by $\frac{\pi}{2}$ using the complex structure of S^2. A more elegant way to establish the interpretation as YM action can be given using differential forms on S^2_N.

[6] the excitations turn out to be monopoles as expected [5], and fluxons similar as in [33]

C. Stability of the vacuum

To establish stability of the vacua (2.16), (2.24) we should work out the spectrum of excitations around this solution and check whether there are flat or unstable modes. This is a formidable task in general, and we only consider the simplest case of the irreducible vacuum (2.16) for the case $\tilde{b} = C_2(N)$ and $d = 0$ here. Once we have established that all fluctuation modes have strictly positive eigenvalues, the same will hold in a neighborhood of this point in the moduli space of couplings $(a, b, d, \tilde{g}, g_6)$.

An intuitive way to see this is by noting that the potential $V(\phi_a)$ can be interpreted as YM gauge theory on S^2_N with gauge group U(n). Since the sphere is compact, we expect that all fluctuations around the vacuum $\phi_a = X_a^{(N)} \otimes 1\!\!1_n$ have positive energy. We fix $n = 1$ for simplicity. Thus we write

$$\phi_a = X_a + A_a(x) \tag{C.1}$$

where $A_a(x)$ is expanded into a suitable basis of harmonics of S^2_N, which we should find. It turns out that a convenient way of doing this is to consider the antihermitian $2N \times 2N$ matrix [5]

$$\Phi = -\frac{i}{2} + \phi_a \sigma_a = \Phi_0 + A \tag{C.2}$$

which satisfies

$$\Phi^2 = \phi_a \phi_a - \frac{1}{4} + \frac{i}{2} \varepsilon_{abc} F_{bc} \sigma_a. \tag{C.3}$$

Thus $\Phi^2 = -\frac{N^2}{4}$ for $A = 0$, and in general we have

$$\tilde{S}_{YM} := Tr(\Phi^2 + \tilde{b} + \frac{1}{4})^2 = Tr\Big((\phi_a \phi_a + \tilde{b})^2 + F_{ab}^\dagger F_{ab}\Big). \tag{C.4}$$

The following maps turn out to be useful:

$$\mathcal{D}(f) := i\{\Phi_0, f\}, \qquad \mathcal{J}(f) := [\Phi_0, f] \tag{C.5}$$

for any matrix f. The maps $\mathcal{D}$ and $\mathcal{J}$ satisfy

$$\mathcal{J}\mathcal{D} = \mathcal{D}\mathcal{J} = i[\Phi_0^2, .], \qquad \mathcal{D}^2 - \mathcal{J}^2 = -2\{\Phi_0^2, .\}, \tag{C.6}$$

which for the vacuum under consideration become

$$\mathcal{J}\mathcal{D} = \mathcal{D}\mathcal{J} = 0, \qquad \mathcal{D}^2 - \mathcal{J}^2 = N^2, \qquad \mathcal{J}^3 = -N^2\, \mathcal{J}. \tag{C.7}$$

Note also that

$$\mathcal{J}^2(f) = [\phi_a, [\phi_a, f]] =: -\Delta f \tag{C.8}$$

is the Laplacian, with eigenvalues $\Delta f_l = l(l+1) f_l$ (for the vacuum).

It turns out that the following is a natural basis of fluctuation modes:

$$\begin{aligned}
\delta\Phi^{(1)} &= A_a^{(1)} \sigma_a = \mathcal{D}(f) - f, \\
\delta\Phi^{(2)} &= A_a^{(2)} \sigma_a = \mathcal{J}^2(f') - \mathcal{J}^2(f')_0 = \mathcal{J}^2(f') + \Delta f' \\
\delta\Phi^{(g)} &= A_a^{(g)} \sigma_a = \mathcal{J}(f'')
\end{aligned} \tag{C.9}$$

for antihermitian $N \times N$ matrices f, f', f'', which will be expanded into orthonormal modes $f = \sum f_{l,m} Y_{lm}$. Using orthogonality it is enough to consider these modes separately, i.e. $f = f_l = -f_l^\dagger$ with $Tr(f_l^\dagger f_l) = 1$. One can show that these modes form a complete set of fluctuations around Φ_0 (for the vacuum). Here $A_{(g)}$ corresponds to gauge transformations, which we will omit from now on. Using

$$Tr(f\mathcal{J}(g)) = -Tr(\mathcal{J}(f)g), \qquad Tr(f\mathcal{D}(g)) = Trf(\mathcal{D}(f)g) \tag{C.10}$$

we can now compute the inner product matrix $TrA^{(i)}A^{(j)}$:

$$\begin{aligned} Tr(A^{(1)}A^{(1)}) &= Tr(((N^2-1)f - \Delta(f))g), \\ Tr(A^{(1)}A^{(2)}) &= Tr(\Delta(f)g), \\ Tr(A^{(2)}A^{(2)}) &= Tr((N^2\Delta(f) - \Delta^2 f))g). \end{aligned} \tag{C.11}$$

It is convenient to introduce the matrix of normalizations for the modes $A^{(i)}$,

$$G_{ij} \equiv \mathrm{Tr}((A^{(i)})^\dagger A^{(j)}) = \begin{pmatrix} (N^2-1) - \Delta, & \Delta \\ \Delta, & N^2\Delta - \Delta^2 \end{pmatrix} \tag{C.12}$$

which is positive definite except for the zero mode $l = 0$ where $A^{(2)}$ is not defined.

We can now expand the action (B.1) up to second order in these fluctuations. Since $F_{ab} = 0$ and $(\phi_a\phi_a + \tilde{b}) = 0$ for the vacuum, we have[7]

$$\delta^2 S_{YM} = Tr\Big(-\frac{1}{\tilde{g}^2}\,\delta F_{ab}\delta F_{ab} + a^2\delta(\phi_a\phi_a)\delta(\phi_b\phi_b)\Big). \tag{C.13}$$

If $a^2 \geq \frac{1}{\tilde{g}^2}$, this can be written as

$$\begin{aligned} \delta^2 S_{YM} &= Tr\Big(\frac{1}{\tilde{g}^2}\,(-\delta F_{ab}\delta F_{ab} + a^2\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)) + (a^2 - \frac{1}{\tilde{g}^2})\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)\Big) \\ &= Tr\Big(\frac{1}{\tilde{g}^2}\,\delta\Phi^2\delta\Phi^2 + (a^2 - \frac{1}{\tilde{g}^2})\delta(\phi_a\phi_a)\delta(\phi_a\phi_a)\Big) \end{aligned} \tag{C.14}$$

and similarly for $a^2 < \frac{1}{\tilde{g}^2}$. It is therefore enough to show that

$$\delta^2 \tilde{S}_{YM} = Tr(\delta\Phi^2\delta\Phi^2) = \mathrm{Tr}(-\delta^{(i)}F_{ab}\delta^{(j)}F_{ab} + \delta^{(i)}(\phi\cdot\phi)\delta^{(j)}(\phi\cdot\phi)) \tag{C.15}$$

has a finite gap in the excitation spectrum. This spectrum can be computed efficiently as follows: note first

$$\begin{aligned} \delta^{(1)}\Phi^2 &= -i\mathcal{D}^2(f) + i\mathcal{D}(f) = -i\mathcal{J}^2(f) + i\mathcal{D}(f) - iN^2 f, \\ \delta^{(2)}\Phi^2 &= -i\mathcal{D}(\Delta f), \\ \delta^{(g)}\Phi^2 &= -i\mathcal{D}\mathcal{J}(f) = [\Phi_0^2, f] = 0 \end{aligned} \tag{C.16}$$

[7]Note that $\delta Tr(\phi\cdot\phi) = 0$ except for the zero mode $A_0^{(1)}$ with $l = 0$ where $\delta^{(1)}Tr(\phi\cdot\phi) \neq 0$, as follows from (C.16). This mode corresponds to fluctuations of the radius, which will be discussed separately.

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for the vacuum. One then finds

$$\begin{aligned} Tr(\delta^{(1)}(\Phi^2)\delta^{(1)}(\Phi^2)) &= -Tr(f)((-(N^2+1)\Delta + (N^2-1)N^2)g), \\ Tr(\delta^{(1)}(\Phi^2)\delta^{(2)}(\Phi^2)) &= -Tr(f)(\Delta^2)(g), \\ Tr(\delta^{(2)}(\Phi^2)\delta^{(2)}(\Phi^2)) &= -Tr(g)(-\Delta^3 + N^2\Delta^2)g). \end{aligned} \tag{C.17}$$

Noting that the antihermitian modes satisfy $Tr(f_l f_l) = -1$, this gives

$$\delta^2 \tilde{S}_{YM} = \begin{pmatrix} -(N^2+1)\Delta + N^4 - N^2, & \Delta^2 \\ \Delta^2, & -\Delta^3 + N^2\Delta^2 \end{pmatrix} = GT \tag{C.18}$$

where the last equality defines T. The eigenvalues of T are found to be N^2 and Δ. These eigenvalues coincide[8] with the spectrum of the fluctuations of $\tilde{S}_{YM}$. In particular, all modes with $l > 0$ have positive mass. The $l = 0$ mode

$$A_0^{(1)} = \mathcal{D}(f_0) - f_0 = (2i\Phi_0 - 1)f_0 = 2if_0\,\sigma_a\phi_a \tag{C.19}$$

requires special treatment, and corresponds precisely to the fluctuations of the normalization α, i.e. the radius of the sphere. We have shown explicitly in (3.13) that this $\alpha = \alpha(y)$ has a positive mass. Therefore we conclude that all modes have positive mass, and there is no flat or unstable direction. This establishes the stability of this vacuum.

The more general case $\tilde{b} = C_2(N) + \epsilon$ with $\alpha \neq 1$ could be analyzed with the same methods, which however will not be done in this paper. For the reducible vacuum (2.24) or (2.26) the analysis is more complicated, and will not be carried out here.

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[8] To see this, assume that we use an orthonormal basis $A^o_{(i)}$ instead of the basis (C.9), i.e. $A = b_1 A^o_{(1)} + b_2 A^o_{(2)}$. Then we can write $G = g^T g$ and $b_i = g_{ij} a_j$. Thus (C.18) becomes $a^T\ GT\ a = b^T\ g\ T\ g^{-1} b$, and the eigenvalues of $g\ T\ g^{-1}$ coincide with those of T, which therefore gives the masses.

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